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Journal of APPLIED MATHEMATICS AND MECHANICS

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Journal of Applied Mathematics and Mechanics 68 (2004) 893–897

THE STABILITY AND BRANCHING OF THE PERMANENT ROTATIONS OF A RIGID BODY WITH A FLUID FILLING†

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(*Received* 27 *November* 2003)

The problem of the motion of a dynamically symmetrical heavy body with a fixed point and an axially symmetrical cavity completely filled with a fluid is considered taking account of internal friction. All the permanent rotations of the system are found and their stability and branching are investigated. The results are presented in the form of an atlas of bifurcation diagrams.

Basic results in the dynamics of a rigid body with cavities containing a fluid have been described in [1]. A phenomenological model of internal viscous friction was proposed in [2] and this model was tested in linear problems of the description of small oscillations in the neighbourhood of uniform rotations of a system around a vertically situated axis of symmetry. This model is used below to construct bifurcation diagrams of the steady motions of a symmetrical heavy body with a viscous filler (a non-linear problem) which, together with the above-mentioned motions, includes uniform rotations around the vertical when the axis of symmetry of the body is in an inclined position.

1. THE EQUATIONS OF MOTION

Consider the problem of the motion of a heavy rigid body with a fixed point and a cavity which is completely filled with a homogeneous fluid. We will assume that the body is dynamically symmetrical and that the cavity is an ellipsoid of revolution, the axis of symmetry of which coincides with the axis of dynamic symmetry of the body on which the centre of mass of the body is located. In addition, we will assume that the fluid executes simple [1, 2] motion and that friction occurs when the fluid interacts with the walls of the cavity, which depends linearly on the difference between the angular velocity of the body and half the vortex vector of the fluid.

The equations of motion of the body–fluid system have the form (compare with the model considered earlier in [2])

$$A\dot{\boldsymbol{\omega}} + B\dot{\boldsymbol{\Omega}} + [\boldsymbol{\omega}, (A\boldsymbol{\omega} + B\boldsymbol{\Omega})] = [\boldsymbol{\gamma}, \partial V/\partial\boldsymbol{\gamma}] \quad (1.1)$$

$$B\dot{\boldsymbol{\Omega}} + [(\boldsymbol{\omega} + \boldsymbol{\Omega}), C\boldsymbol{\Omega}] = D(\boldsymbol{\omega} - \boldsymbol{\Omega}) \quad (1.1)$$

$$\dot{\boldsymbol{\gamma}} + [\boldsymbol{\omega}, \boldsymbol{\gamma}] = 0 \quad (1.3)$$

†*Prikl. Mat. Mekh.* Vol. 68, No. 6, pp. 994–998, 2004.

0021–8928/$—see front matter.
doi: 10.1016/j.jappmathmech.2004.11.011

Here

$$A = J + J^* = \mathrm{diag}(A_1, A_1, A_3), \quad B = I - J^* = \mathrm{diag}(B_1, B_1, B_3)$$

$$C = \mathrm{diag}(B_1, B_1, 4\delta^2(1+\delta^2)^{-1}B_3), \quad D = \mathrm{diag}(D_1, D_1, D_3), \quad \delta = a_1/a_3$$

$$J = \mathrm{diag}(J_1, J_1, J_3), \quad J^* = \mathrm{diag}(J_1^*, J_1^*, J_3^*), \quad I = \mathrm{diag}(I_1, I_1, I_3)$$

$\boldsymbol{\omega}$ is the angular velocity, $\boldsymbol{\Omega}$ is half the vortex vector, $\boldsymbol{\gamma}$ is the unit vector of the ascending vertical, $V = mgs\gamma_3$ is the potential energy of the system (m is the mass of the whole system, g is the acceleration due to gravity, s is the distance from the fixed point to the centre of mass of the system), J is the inertia tensor of the rigid body for a fixed point referred to the principal axes of inertia of the body, J^* is the inertia tensor of an equivalent body (a_1, $a_2 = a_1$ and a_3 are the semi-axes of the cavity) and I is the central inertia tensor of the fluid, referred to the principal axes of the cavity.

The tensor A is the sum of the inertia tensor of the body and the tensor of the equivalent body. Tensor B is the difference between the inertia tensor of the body and the tensor of the equivalent body, tensor C is of an auxiliary nature and tensor D characterizes the intensity of the internal friction ($(Du, u) > 0$, $\forall u \neq 0$). If the mass of the body is denoted by $m\varepsilon$ and the mass of the fluid is denoted by $m(1-\varepsilon)$, where $\varepsilon \in (0, 1)$, then

$$J_1 = m\varepsilon(\rho_1^2 + s^2), \quad J_3 = m\varepsilon\rho_3^2, \quad J_1^* = \frac{m(1-\varepsilon)}{5}a_3^2\frac{(1-\delta^2)^2}{1+\delta^2}, \quad J_3^* = 0$$

$$I_1 = \frac{m(1-\varepsilon)}{5}a_3^2(1+\delta^2), \quad I_3 = \frac{2m(1-\varepsilon)}{5}a_3^2\delta^2$$

(ρ_1, $\rho_2 = \rho_1$ and ρ_3 are the central radii of inertia of the body).

Equations (1.1) express the change in the angular momentum of the system, Eqs (1.2) describe the evolution of the vortex vector and Eqs (1.3) denote the constancy of the vector $\boldsymbol{\gamma}$ in the absolute frame of reference.

The system of equations (1.1)–(1.3) has been investigated by Thomson, Poincaré, Zhukovskii and others in the case when $D = 0$ (see the detailed bibliography in [1, 3] and the All-Union Institute for Scientific and Technical Information (VINITI) reviews).

2. THE EFFECTIVE POTENTIAL

It follows from the system of equations (1.1)–(1.3) that

$$\frac{dH}{dt} = -(D(\boldsymbol{\omega} - \boldsymbol{\Omega}), (\boldsymbol{\omega} - \boldsymbol{\Omega})) \leq 0, \quad \frac{dK}{dt} = 0$$

where H is the total mechanical energy of the system, and K is the projection of the angular momentum of the system onto the vertical

$$H = \frac{1}{2}[(A\boldsymbol{\omega}, \boldsymbol{\omega}) + (B\boldsymbol{\Omega}, \boldsymbol{\Omega})] + V, \quad K = ((A\boldsymbol{\omega} + B\boldsymbol{\Omega}), \boldsymbol{\gamma})$$

The system being considered therefore allows of a non-increasing function H and a first integral $K = k = \text{const}$.

We find the effective potential [4] of the system as the minimum of the function H with respect to the variables $\boldsymbol{\omega}$ and $\boldsymbol{\Omega}$ for a fixed level of the integral $K = k$. For this purpose we consider the function $F = H - \lambda(K - k)$, where λ is an undetermined Lagrange multiplier and we write out the conditions for its stationarity with respect to the variables ω, $\boldsymbol{\Omega}$ and λ

$$\frac{\partial F}{\partial \boldsymbol{\omega}} = A(\boldsymbol{\omega} - \lambda\boldsymbol{\gamma}) = 0, \quad \frac{\partial F}{\partial \boldsymbol{\Omega}} = B(\boldsymbol{\Omega} - \lambda\boldsymbol{\gamma}) = 0, \quad \frac{\partial F}{\partial \lambda} = k - K = 0 \tag{2.1}$$

The relations $\boldsymbol{\omega} = \lambda\boldsymbol{\gamma}$ and $\boldsymbol{\Omega} = \lambda\boldsymbol{\gamma}$ follow from the first two equations of system (2.1), and, on substituting these into the last equation of this system, we find that

$$\lambda = k/P(\boldsymbol{\gamma}), \quad P(\boldsymbol{\gamma}) = P_1(\gamma_1^2 + \gamma_2^2) + P_3\gamma_3^2; \quad P_1 = A_1 + B_1, \quad P_3 = A_3 + B_3$$

Hence, a minimum of the function H at the level $K = k$ is attained when

$$\boldsymbol{\omega} = \frac{k}{P(\boldsymbol{\gamma})}\boldsymbol{\gamma}, \quad \boldsymbol{\Omega} = \frac{k}{P(\boldsymbol{\gamma})}\boldsymbol{\gamma} \tag{2.2}$$

and is equal to

$$W_k(\boldsymbol{\gamma}) = mgs\gamma_3 + \frac{k^2}{2P(\boldsymbol{\gamma})} \tag{2.3}$$

According to Routh's theory, the steady motions of the system

$$\boldsymbol{\omega} = \frac{k}{P(\boldsymbol{\gamma}_0)}\boldsymbol{\gamma}_0, \quad \boldsymbol{\Omega} = \frac{k}{P(\boldsymbol{\gamma}_0)}\boldsymbol{\gamma}_0, \quad \boldsymbol{\gamma} = \boldsymbol{\gamma}_0 \tag{2.4}$$

correspond to the critical points $\boldsymbol{\gamma} = \boldsymbol{\gamma}_0$ of the function $W_k(\boldsymbol{\gamma})$ on the Poisson sphere $\boldsymbol{\gamma}^2 = 1$. These steady motions are obviously permanent rotations of the system as a rigid body ($\boldsymbol{\omega} = \boldsymbol{\Omega}$) about a vertical, while the stable steady motions correspond to minimum points.

We will show that the function H decreases for all motions which differ from the steady motions (2.4). Actually, $dH/dt \equiv 0$ if and only if $\boldsymbol{\omega} \equiv \boldsymbol{\Omega}$; in this case (see 1.2)) $\dot{\boldsymbol{\Omega}} \equiv 0$, that is, $\dot{\boldsymbol{\omega}} \equiv 0$. Hence, $\dot{\gamma}_3 \equiv 0$ (see the expression for the function H) and $\dot{\gamma}_1 \equiv \dot{\gamma}_2 \equiv 0$ (see the projection of Eq (1.1) onto the two first principal axes of inertia of the body). Consequently, the partially asymptotically stable motions correspond to the minima of the effective potential and the unstable motions correspond to the other critical points. In the first case, partial asymptotic stability means that the perturbed motion tends to a permanent rotation (but not necessarily to unperturbed motion).

3. STABILITY OF THE TRIVIAL STEADY MOTIONS

In order to investigate the critical points of the effective potential on the Poisson sphere, we reduce it to the form

$$W_k = mgsf(\gamma_3), \quad f(\gamma_3) = \gamma_3 + \frac{\kappa^2}{2[\gamma_3^2 + p(1-\gamma_3^2)]}; \quad \kappa^2 = \frac{k^2}{mgsP_3}, \quad p = \frac{P_1}{P_3}$$

It is obvious that the function $f(\gamma_3)$ is defined in the interval $\gamma_3 \in [-1, 1]$ and always has the critical points $\gamma_3 = -1$ and $\gamma_3 = 1$. The trivial uniform rotations around the vertically positioned axis of symmetry correspond to these points:

$$\gamma_1 = \gamma_2 = 0, \quad \omega_1 = \omega_2 = 0, \quad \Omega_1 = \Omega_2 = 0, \quad \gamma_3 = -1, \quad \omega_3 = \Omega_3 = \omega = -k/P_3 \tag{3.1}$$

$$\gamma_1 = \gamma_2 = 0, \quad \omega_1 = \omega_2 = 0, \quad \Omega_1 = \Omega_2 = 0, \quad \gamma_3 = 1, \quad \omega_3 = \Omega_3 = \omega = k/P_3 \tag{3.2}$$

in the case of the lowest position (3.1) or highest position (3.2) of the centre of mass.

The first rotations are stable (unstable) if $f'(-1) > 0$ (<0) and the second rotations are stable (unstable) if $f'(1) < 0$ (>0). On calculating $f' = df/d\gamma_3$, we have

$$f' = 1 - \frac{\kappa^2(1-p)\gamma_3}{[\gamma_3^2 + p(1-\gamma_3^2)]^2}$$

The conditions for the stability (instability) of the uniform rotations (3.1) and (3.2) therefore, respectively, have the form

$$1 + \kappa^2(1-p) > 0 \ (<0), \quad 1 - \kappa^2(1-p) < 0 \ (>0)$$

or

$$mgs + (A_3 + B_3 - A_1 - B_1)\omega^2 > 0 \ (<0) \tag{3.3}$$

$$mgs - (A_3 + B_3 - A_1 - B_1)\omega^2 < 0 \ (>0) \tag{3.4}$$

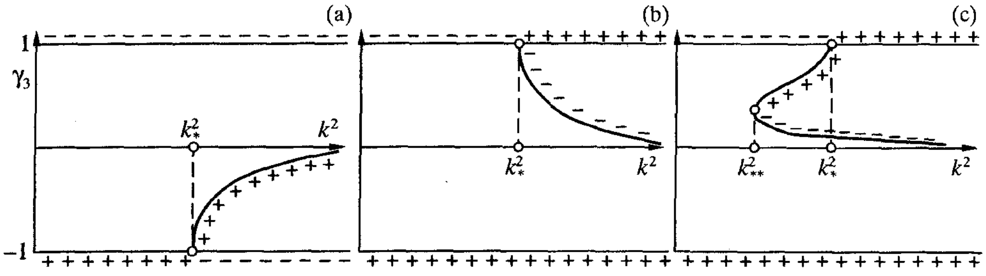

Fig. 1

Consequently, the uniform rotations (3.1) are always stable (see (3.3) when $A_3 + B_3 > A_1 + B_1$ (that is, when $p < 1$) and the uniform rotations (3.2) are always unstable (see (3.4)) when $p > 1$.

When $p > 1$, the uniform rotations (3.1) are stable when $\kappa^2 < \kappa_*^2$ ($\omega^2 < \omega_*^2$) and, when $p < 1$, the uniform rotations (3.2) are stable when $\kappa^2 > \kappa_*^2$ ($\omega^2 > \omega_*^2$). Here

$$\kappa_*^2 = \frac{p^2}{|1-p|}, \quad \omega_*^2 = \frac{mgs}{|A_3 + B_3 - A_1 - B_1|}$$

Hence, when $p > 1$ ($p < 1$) a change occurs in the stability of the uniform rotations (3.1) ((3.2)) and "skew" ($\gamma_3^2 \neq 1$, that is, $\gamma_1^2 + \gamma_2^2 \neq 0$) permanent rotations must branch off from them.

It can be seen that conditions (3.3) and (3.4) do not contain elements of the tensor D. This fact correlates fully with the result obtained by Rumyantsev [1]: the conditions for the stability of the permanent rotation of a body with a viscous filler when there is no external dissipation do not contain the coefficient of viscosity of the filler. This property is explained by the fact that conditions (3.3) and (3.4) separate out the domains of "secular stability". In the case when $D = 0$, the domain of "gyroscopic stabilization", which is known to be destroyed in the case of dissipative forces which may be as small as desired, can be added to this domain.

The magnitude of the viscosity affects the rate of the transients, to estimate which system (1.1)–(1.3) can be used.

4. BRANCHING OF STEADY MOTIONS

In order to find the "skew" permanent rotations, we consider the equation $f' = 0$ from which the internal ($\gamma_3 \in (-1, 1)$) critical points of the effective potential are determined and we represent it in the form

$$\kappa^2 = \frac{[\gamma_3^2 + p(1-\gamma_3^2)]^2}{(1-p)\gamma_3} = \varphi(\gamma_3) \tag{4.1}$$

It is obvious that Eq. (4.1) when $p > 1$ can only have negative roots ($\gamma_3 \in (-1, 0)$) and, when $p < 1$, it can only have positive roots ($\gamma_3 \in (0, 1)$). Note that this equation is identical, apart from the notation, with the equation from which the "skew" steady motions of a physical pendulum, suspended from a horizontal axis which can freely rotate about a vertical (see [4, 5]), are determined. Analysis of the function $\varphi(\gamma_3)$ (see also [4, 5]) shows that, when $p > 1$ and $3/4 < p < 1$, this function is monotonic and, when $p \in (0, 3/4)$, there is just a single extremal point: $\gamma_3 = \sqrt{p/(3(p-1))}$. Hence, all the critical points of the function $f(\gamma_3)$ can be represented in the form of the curves $\gamma_3 = \gamma_3(k^2)$ in the (k^2; γ_3) plane. Here we must distinguish three cases: (a) $p > 1$, (b) $3/4 < p < 1$ and (c) $0 < p < 3/4$. These curves are shown in Fig. 1, where

$$k_*^2 = mgsP_3\frac{p^2}{|1-p|}, \quad k_{**}^2 = k_*^2\frac{4}{3\sqrt{3}}\frac{p\sqrt{p}}{\sqrt{1-p}}$$

Stable (unstable) permanent rotations are labelled with plus (minus) signs, and conclusions concerning the stability or instability of the "skew" permanent rotations ($\gamma_3 \neq \pm 1$) are drawn on the basis of the theory of bifurcations [4, 5].

5. EXAMPLES

As an example, we will consider the case of a heavy thin-walled spheroid filled with a fluid and clamped at a vertex lying on its axis of symmetry. In this case,

$$s = a_3, \quad A_1 = \frac{ma_3^2}{3}\varepsilon(1+\delta^2) + ma_3^2\varepsilon = \frac{ma_3^2}{3}\varepsilon(4+\delta^2)$$

$$A_3 = \frac{ma_3^2}{3}\varepsilon 2\delta^2, \quad B_1 = \frac{4}{5}ma_3^2(1-\varepsilon)\frac{\delta^2}{1+\delta^2}, \quad B_3 = \frac{2}{5}ma_3^2(1-\varepsilon)\delta^2$$

Hence,

$$p = \frac{12\delta^2 + \varepsilon(5\delta^4 + 13\delta^2 + 20)}{2\delta^2(1+\delta^2)(3+2\varepsilon)}$$

and, if $\delta^2 < 1$ (the prolate spheroid), then $p > 1$ for any $\varepsilon \in (0, 1)$ (that is, for any ratio of the mass of the shell to the mass of the whole system). If, however, $\delta^2 > 1$, then $p \gtrless 1$ when $\varepsilon \gtrless \varepsilon_1$ and $p \gtrless 3/4$ when $\varepsilon \gtrless \varepsilon_2$, where

$$\varepsilon_1 = \frac{6\delta^2(\delta^2-1)}{(\delta^2+4)(\delta^2+5)}, \quad \varepsilon_2 = \frac{3}{4}\frac{\delta^2(3\delta^2-5)}{\delta^4+5\delta^2+10}$$

(it is obvious that $\varepsilon_2 < 0$ if $\delta^2 < 5/3$).

Consequently, if the cavity is a prolate spheroid ($a_1 < a_3$), then, for any ratio of the mass of the shell to the mass of the whole system, the bifurcation diagram has the form shown in Fig. 1(a). If, however, the cavity is an oblate spheroid ($a_1 > a_3$), the form of the bifurcation diagram depends on the cavity aspect ratio and on the ratio of the mass of the shell to the mass of the whole system: if the cavity is not very strongly oblate ($1 < a_1/a_3 < \sqrt{5/3}$), the bifurcation diagram has the form shown in Fig. 1(a) when $\varepsilon > \varepsilon_1$ (the shell is fairly heavy) or in Fig. 1(b) when $\varepsilon < \varepsilon_1$ (the shell is quite light). If, however, the cavity is strongly oblate ($a_1/a_3 > \sqrt{5/3}$), the bifurcation diagram has the form shown in Fig. 1(a) when $\varepsilon > \varepsilon_1$, in Fig. 1(b) when $\varepsilon \in (\varepsilon_2, \varepsilon_1)$ and in Fig. 1(c) when $\varepsilon < \varepsilon_2$.

In particular, in the case of an almost weightless shell ($\varepsilon = +0$), the bifurcation diagram has the form a, b or c when $\delta^2 < 1$, $1 < \delta^2 < 5/3$, or $\delta^2 > 5/3$ respectively. If, however, the fluid is very light ($\varepsilon = 1-0$), then the bifurcation diagram has the form a, b or c when $\delta^2 < 4$, $4 < \delta^2 < 8$ or $\delta^2 > 8$ respectively.

This research was partially supported financially by the Russian Foundation for Basic Research (04-01-00398 and 03-01-00190) and the Federal Special Purpose "Integration" Programme (A0097).

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Translated by E.L.S.